Nonlocal approach to the analysis of the stress distribution in granular systems. I. Theoretical framework

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A theoretical framework for the analysis of the stress distribution in granular materials is presented. It makes use of a transformation of the vertical spatial coordinate into a formal time variable and the subsequent study of a generally non-Markoffian, i.e., memory-possessing (nonlocal) propagation equation. Previous treatments are obtained as particular cases corresponding to, respectively, wavelike and diffusive limits of the general evolution. Calculations are presented for stress propagation in bounded and unbounded media. They can be used to obtain desired features such as a prescribed stress distribution within the compact.
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I. INTRODUCTION

Considerable activity is apparent in the recent literature [1–15] in the calculation of the stress distribution in granular materials ranging from sandpiles to ceramic powder compacts. The interest stems in part from new experimental results [16–18], in part from the fact that important features of old experimental data [19–21] have never been satisfactorily explained, and in part from the need to understand and control the formation of undesirable density gradients in the manufacture of metal and ceramic parts. Variations in the density distribution have been identified as a source of shrinking, cracking, and failure during the pressing and sintering processes [22–24].

Our purpose in the present paper is to provide a theoretical framework for the analysis of stress distribution in granular materials on the basis of the idea of the formal interpretation of the vertical spatial coordinate as time. Such an idea appears in two earlier analyses in the literature. One of them is by Bouchaud *et al.* [8] who use continuum mechanics to derive a ''wave equation'' to describe the ''coherent'' transmission of stress in a granular compact. The other, although it might not have been recognized as such, is the quite unrelated analysis of Liu *et al.* [11] who describe stress distribution through the use of a discrete Master equation following what may be regarded as a Markoffian evolution. Our approach in the present paper starts in the continuum mechanics picture, proceeds through the introduction of formally natural constitutive relations more general than those used in earlier work, and ends in propagation equations that are *nonlocal* in the vertical coordinate. A simple case of our equation turns out to be identical to the telegrapher's equation [25]. With its help, we are able to unify the treatments of Bouchaud *et al.* [8] and of Liu *et al.* [11] by showing that, while seemingly disparate, they are extreme consequences of our general treatment. To the best of our knowledge, this interconnection has not been realized earlier. It allows one to combine excellent nonoverlapping insights developed into stress transmission by the authors of those two references. Furthermore, our theory describes intermediate situations wherein the transmission of stress is neither completely coherent [8] nor completely incoherent [11] .

II. STRESS BALANCE EQUATIONS AND OUTLINE OF OUR APPROACH

Some features of stress distribution in powder compacts arise directly from their granular nature. Others can be described through an application of continuum mechanics. In the present paper, we start with continuum mechanics as in the analysis of Ref. [8]. We focus attention on the stress tensor, which, in rectangular coordinates, is given by

$$\sigma=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}. \tag{2.1}$$

In the absence of torques, the off-diagonal shear-stress terms are equal in pairs ($\sigma_{xy}=\sigma_{yx}$, $\sigma_{yz}=\sigma_{zy}$, and $\sigma_{zx}=\sigma_{xz}$). Newton's second law of motion is described by the Cauchy relation

$$\nabla\cdot\sigma+\rho\mathbf{b}=\rho\frac{d\mathbf{v}}{dt}, \tag{2.2}$$

where ρ is the mass density of the body at a point, $d\mathbf{v}/dt$ is the acceleration of the point, and $\mathbf{b}$ is the body force per unit mass acting on the point. In equilibrium, if the only body force acting on the point is the gravitational force in the z direction, we have

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}=0, \tag{2.3}$$

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}=0, \tag{2.4}$$

$$\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}=\rho g. \tag{2.5}$$

These stress balance equations describe the behavior of the six unique elements of σ. Because interest lies primarily in the local density in the compact, which is believed to be a function [26,27] of the local value of σ_{zz}, we will focus entirely on σ_{zz}. Equation (2.5), which governs this quantity, can be recast in the form of a two-dimensional continuity equation:

$$\nabla \cdot \mathbf{j}+\frac{\partial S}{\partial z}=\rho g. \tag{2.6}$$

The ''flux'' $\mathbf{j}$ is a two-dimensional vector with σ_{xz} and σ_{yz} as its x and y components, respectively, the ''density'' S of the fictitious fluid whose flow is given by Eq. (2.6) is identical to σ_{zz}, and ρg is a ''source term.'' In this interpretation, the z coordinate assumes the role of time [8]. If the applied stresses are much larger than the gravitational force, the ''source term'' can be neglected. For the sake of simplicity, we will consider this situation realized in the body of the paper and return to the source term in Sec. VI.

Equation (2.6), which we now consider with its right-hand side put equal to zero, can be used to determine S only if an additional equation relating the components of $\mathbf{j}$ to S is given. In order to understand the spirit of our analysis below, it is useful to consider three examples of such an additional equation which are known in fluid flow. The first example is Fick's law:

$$\mathbf{j}=-D\nabla S, \tag{2.7}$$

which, when substituted in the continuity equation, leads to the diffusion equation for S, with D the diffusion constant. The symbol ∇ here represents the *two-dimensional* gradient. The second example is

$$\frac{\partial \mathbf{j}}{\partial z}=-c^2\nabla S, \tag{2.8}$$

and describes the proportionality between the ''time'' derivative (z derivative) of the flux and the gradient of the density. It leads to the wave equation, with wave propagation speed c. The third example combines features of the first and the second examples,

$$\frac{\partial \mathbf{j}}{\partial z}+\frac{c^2}{D}\mathbf{j}=-c^2\nabla S, \tag{2.9}$$

and reduces to the previous two limits in easily understood extreme limits. When substituted in the continuity equation, Eq. (2.9) leads to what is known as the telegrapher's equation [25,28]:

$$\frac{\partial^2 S}{\partial z^2}+\frac{c^2}{D}\frac{\partial S}{\partial z}=c^2\nabla^2 S. \tag{2.10}$$

This telegrapher's equation, which we will find of considerable use in the present paper, unifies diffusive and wave behaviors in a straightforward fashion and is itself a special case of a generalized memory equation [29] where the constitutive relation is nonlocal in the z coordinate

$$\mathbf{j}(z)=-D\int_0^z dz'\,\phi(z-z')\nabla S(z'). \tag{2.11}$$

The resulting equation governing S is an integrodifferential equation of the Volterra type,

$$\frac{\partial S(z)}{\partial z}=D\int_0^z dz'\,\phi(z-z')\nabla^2 S(z'), \tag{2.12}$$

and reduces to the diffusion, the wave, and the telegrapher's equations in the respective limits of a ''memory'' that vanishes [$\phi(z)=\delta(z)$], is constant [$\phi(z)=c^2/D$], and is intermediate [$\phi(z)=(c^2/D)e^{-(c^2/D)z}$].

We will see below that an examination of the constitutive relations commonly assumed in the literature on stress transmission in granular media (e.g., in Ref. [8]) suggests a natural generalization, that the generalization leads to the non-Markoffian evolution equation (2.12) reducing in a simple instance to the telegrapher's equation (2.10), and that it is possible, on the basis of these equations, to construct a detailed framework for the description of stress distribution in granular media.

III. GENERALIZED CONSTITUTIVE (CLOSURE) RELATIONS

Since the stress balance equations (2.3), (2.4), (2.5) are three in number but involve six independent quantities, they cannot be solved unless additional relations are introduced among the six stress tensor components. Such relations are known as constitutive or closure relations. In the present state of the theory of stress distribution in granular materials, they form the weakest link because, whether made explicit or implicit in the analysis, they are *ad hoc* in nature. This is true of all such relations to be found in the literature [8,11,24,30–33]. It appears extremely difficult, at the present stage of the field, to provide any satisfactory physical justification for existing relations. As in the case of other constitutive relations in the literature, the one we propose below is also *ad hoc*. However, it is mathematically natural and has the distinct advantage of leading to a unification and generalization of earlier treatments.

The closure assumption of Janssen [30] and Thompson [24], particularly as expressed by Bouchaud *et al.* [8], postulates proportionality between the diagonal elements of the stress tensor (σ_{xx}, σ_{yy}, and σ_{zz}), as well as the vanishing of the shear components in the x-y plane: $\sigma_{xy}=\sigma_{yx}=0$. In the second part of their analysis, Bouchaud *et al.* [8] attempt to go beyond this assumption through the inclusion of nonlinear, second-order corrections in σ_{xz} and σ_{yz}. Nonlinear constitutive relations also appear in the treatment by Edwards and Mounfield [15].

In the absence of a satisfactory physical argument, the most natural constitutive relation is a simple proportionality between σ_{xx}, σ_{yy}, and σ_{zz}, as assumed in Ref. [8]:

$$\sigma_{xx}=\text{const}\times\sigma_{zz}, \quad \sigma_{yy}=\text{const}\times\sigma_{zz}. \tag{3.1}$$

We note, however, that this relation is never used directly [8] for combining with Eqs. (2.3),(2.4), but only in the form of spatial derivatives:

$$\frac{\partial\sigma_{xx}}{\partial x}=c_1{}^2\frac{\partial\sigma_{zz}}{\partial x},\quad \frac{\partial\sigma_{yy}}{\partial y}=c_2{}^2\frac{\partial\sigma_{zz}}{\partial y}. \tag{3.2}$$

Here c_1^2 and c_2^2 are constants of proportionality that are equal to each other in isotropic media. We develop our constitutive relation by first reexpressing Eq. (3.1) in the derivative form (3.2), and then generalizing it to incorporate the contributions of σ_{xz} and σ_{yz}. We represent these contributions through the addition of first-order terms in the sense of a Taylor's series expansion:

$$\frac{\partial\sigma_{xx}}{\partial x}=c_1{}^2\frac{\partial\sigma_{zz}}{\partial x}+\alpha_1\sigma_{xz},\quad \frac{\partial\sigma_{yy}}{\partial y}=c_2{}^2\frac{\partial\sigma_{zz}}{\partial y}+\alpha_2\sigma_{yz}. \tag{3.3}$$

As in the case of $c_1$and c_2, the quantities α_1 and α_2 would equal each other in isotropic media. Combining these results with Eqs. (2.3) and (2.4) and, assuming, as in the analysis of Ref. [8] that $\sigma_{xy}=\sigma_{yx}=0$, we obtain

$$c_1{}^2\frac{\partial\sigma_{zz}}{\partial x}+\alpha_1\sigma_{xz}+\frac{\partial\sigma_{xz}}{\partial z}=0, \tag{3.4}$$

$$c_2{}^2\frac{\partial\sigma_{zz}}{\partial y}+\alpha_2\sigma_{yz}+\frac{\partial\sigma_{yz}}{\partial z}=0. \tag{3.5}$$

Identifying, as in Eq. (2.6), the components j_x, j_y of the two-dimensional ''flux'' $\mathbf{j}$ with σ_{xz} and σ_{yz}, respectively, and the ''fluid density'' S with σ_{zz}, we express our constitutive relation (3.4), (3.5) as

$$\frac{\partial\mathbf{j}}{\partial z}+\alpha_1 j_x\hat{\mathbf{a}}_x+\alpha_2 j_y\hat{\mathbf{a}}_y=-\left(c_1{}^2\frac{\partial S}{\partial x}\hat{\mathbf{a}}_x+c_2{}^2\frac{\partial S}{\partial y}\hat{\mathbf{a}}_y\right), \tag{3.6}$$

where $\hat{\mathbf{a}}_x$ and $\hat{\mathbf{a}}_y$ are unit vectors in x and y directions, respectively. Equation (3.6) or its isotropic counterpart Eq. (2.9) (which we will use in most of the analysis below) is our constitutive relation. It leads to the telegrapher's equation (2.10) for σ_{zz}. While we have assumed above, following Ref. [8], that σ_{xy} and σ_{yx} are identically equal to zero, it is actually not necessary to make this assumption to get Eq. (3.6). It is enough to postulate that Eq. (3.3) is replaced by

$$\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xx}}{\partial x}=c_1{}^2\frac{\partial\sigma_{zz}}{\partial x}+\alpha_1\sigma_{xz},$$

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=c_2{}^2\frac{\partial\sigma_{zz}}{\partial y}+\alpha_2\sigma_{yz}. \tag{3.7}$$

We close this section by rewriting our constitutive relation (2.9) in the memory form (2.11) with $D=c^2/\alpha$ and $\phi(z)=\alpha\exp(-\alpha z)$:

$$\mathbf{j}(z)=-c^2\int_0^z dz'\,e^{-\alpha(z-z')}\boldsymbol{\nabla}S(z'). \tag{3.8}$$

IV. OUR EVOLUTION EQUATION AND UNIFICATION OF EARLIER TREATMENTS

Our memory form of the constitutive relation (2.11) leads to the nonlocal equation (2.12). The general program of analysis may thus proceed by assuming on physical grounds, or determining from experiment, the memory function $\phi(z)$ and the quantity D, and then solving Eq. (2.12) for σ_{zz}, i.e., S. We write Eq. (2.12) explicitly in the form

$$\frac{\partial\sigma_{zz}(x,y,z')}{\partial z}=D\int_0^z dz'\,\phi(z-z')\nabla^2\sigma_{zz}(x,y,z'). \tag{4.1}$$

Properties of the granular material would be reflected in $\phi(z)$ and D, and complex behavior could be described through appropriate forms of $\phi(z)$. A particularly useful feature of the theory we present is the unification it provides of two seemingly unrelated treatments of stress distribution available in the literature. The connection of our theory to that of Bouchaud *et al.* [8] is obvious, given that we have followed their analysis closely in developing the present treatment. We simply take the memory function in Eq. (4.1) to be a constant: $\phi(t)=c^2/D$. We then obtain, from our Eq. (4.1), the wave equation

$$\frac{\partial^2\sigma_{zz}(x,y,z)}{\partial z^2}=c^2\nabla^2\sigma_{zz}(x,y,z)=c^2\left[\frac{\partial^2\sigma_{zz}(x,y,z)}{\partial x^2}+\frac{\partial^2\sigma_{zz}(x,y,z)}{\partial y^2}\right], \tag{4.2}$$

which, with the identification of c^2 with k_1, is the essential result of Ref. [8]. The interpretation here is that the spatial memory function corresponding to the analysis of Ref. [8] describes the perfect retention of information on how stress propagates from layer to layer in the z direction. Such a system can be imagined as consisting of identical, frictionless spherical particles arrayed in a perfectly ordered lattice. The stress applied on one particle would be transmitted along the lines of contact between particles and there would be no loss of information about the original strength and direction of the applied force.

This coherent limit does not fully describe a realistic granular system in which random-shaped particles of random sizes are packed in a random arrangement. This would suggest that stress propagation might be best described in terms of Markoffian processes characteristic of spatial memories of the system that decay quickly with the result that the stress path takes on the behavior of a random walk in the medium. The limit of our theory, which is opposite to that in which we recover the results of Bouchaud *et al.* [8] as discussed above, is one in which the memory is not perfect (constant, nondecaying) but decays immediately, i.e., $\phi(z)=\delta(z)$. In such a limit we recover the equation

$$\frac{\partial\sigma_{zz}(x,y,z)}{\partial z}=D\nabla^2\sigma_{zz}(x,y,z)$$
$$=D\left[\frac{\partial^2\sigma_{zz}(x,y,z)}{\partial x^2}+\frac{\partial^2\sigma_{zz}(x,y,z)}{\partial y^2}\right]. \tag{4.3}$$

We shall now show that this result (4.3) is, in essence, the evolution equation of Liu *et al.* [11].

In the analysis of Liu *et al.* [11], the transmitted stress per granular particle, $w(z,x)$, is described as arising from a sum of contributions from random probabilistic transmission of forces from particles in one layer of the granular material to particles in the next lower layer. The system is discretized in terms of layers of a given thickness d. (We will denote the thickness by the symbol d instead of by D used in Ref. [11] to avoid confusion with our ''diffusion constant''.) Sites j, i are considered to be in the horizontal direction and a sum is taken over particles in the previous layer that participate in the transmission of forces to the jth particle. With $q_{ij}(d)$ as the random fraction of the stress that passes from the ith site in layer d to the jth site in layer $d+1$, the evolution of the stress is described [Eq. (2) of Ref. [11]] through

$$w(d+1,j)=m(d+1,j)g+\sum_i q_{ij}(d)w(d,i). \tag{4.4}$$

We have shown explicitly here the weight $m(d+1,j)g$, the product of the mass of the particle in layer $d+1$ at site j and the acceleration due to gravity. In their analysis, the authors of Ref. [11] take that source term to be unity. If we neglect this term under the standard assumption that the applied pressure is much greater than the internal stresses due to gravity, notice that the force transmitted through the particle j on layer $d+1$ must equal the total of the fractional force transmitted through the particles i on layer d, i.e., $\Sigma_i q_{ij}(d)=1$ [see Eq. (3) of Ref. [11]], and subtract from Eq. (4.4) the identity

$$w(d,j)=\left[\sum_i q_{ij}(d)\right]w(d,j), \tag{4.5}$$

we find

$$w(d+1,j)-w(d,j)=\sum_i [q_{ij}(d)w(d,i)-q_{ji}(d)w(d,j)]. \tag{4.6}$$

This is a difference equation in the discrete layer variable d whose increment Δd equals 1. Going to the continuum limit in the variable d, dividing by Δd, and taking the limit $\Delta d\to 0$, we obtain the well-known Master equation

$$\frac{\partial w(z,j)}{\partial z}=\sum_i [F_{ij}(z)w(z,i)-F_{ji}(z)w(z,j)], \tag{4.7}$$

where $F_{ij}(z)$ is $\lim_{\Delta d\to 0} q_{ij}(d)/\Delta d$ and we have replaced the discrete variable d by the continuous variable z. If, for simplicity, we take the F's to be nearest neighbor in extent and independent of location within the layer, and the horizontal space to be one dimensional, we have

$$\frac{\partial w(z,j)}{\partial z}=F(z)[w(z,j+1)+w(z,j-1)-2w(z,j+1)], \tag{4.8}$$

which, in the continuum limit in j, reduces to the diffusion equation with a diffusion constant that is dependent on both z and x,y. More generally, if we take the quantities F_{ij} to be local in the x-y space, the right-hand side can be written as proportional to the two-dimensional Laplacian operator. With the identification of w with σ_{zz}, we thus have Eq. (4.3), which is the extreme Markoffian limit of our theory, as representing the theory of Liu *et al.* [11].

The authors of Refs. [8] and [11] appear not to have made use of each other's work and to have proceeded from their evolution equations in entirely different ways. We hope that the unification we have provided here will allow further work in this field to combine insights offered by both sets of analysis.

V. APPLICATIONS OF THE THEORY

As illustrations of the usefulness of our theoretical framework, we work out its consequences for stress distribution in unbounded media and in long pipes, respectively. We will use for this purpose the telegrapher's equation

$$\frac{\partial^2\sigma_{zz}(x,y,z)}{\partial z^2}+\alpha\frac{\partial\sigma_{zz}(x,y,z)}{\partial z}=c^2\nabla^2\sigma_{zz}(x,y,z) \tag{5.1}$$

with $D=c^2/\alpha$. For the sake of simplicity, we will consider here only a two-dimensional system and thus use, instead of Eq. (5.1), the equation

$$\frac{\partial^2\sigma_{zz}(x,z)}{\partial z^2}+\alpha\frac{\partial\sigma_{zz}(x,z)}{\partial z}=c^2\frac{\partial^2\sigma_{zz}(x,z)}{\partial x^2}. \tag{5.2}$$

A. Stress distribution in unbounded media

We take the applied stress $\sigma_{zz}(x,0)$ at the ''surface'' $z=0$ to be a delta function $\delta(x)$. The solution of Eq. (5.2) is then given by

$$\sigma_{zz}(x,z)=e^{-\alpha z/2}\left[\frac{\delta(x+cz)+\delta(x-cz)}{2}+T\right], \tag{5.3}$$

where the term T vanishes identically for $cz\leqslant x$, and equals, for $cz\geqslant x$,

$$T=\left(\frac{\alpha}{4c}\right)\left\{I_0\left(\frac{\alpha}{2c}\sqrt{c^2z^2-x^2}\right)+\frac{cz}{\sqrt{c^2z^2-x^2}}I_1\left(\frac{\alpha}{2c}\sqrt{c^2z^2-x^2}\right)\right\} \tag{5.4}$$

the I's being modified Bessel functions. We immediately recover the interesting phenomenon of ''light cones'' discovered by Bouchaud *et al.* [8]. Thus, in the limit $\alpha=0$,

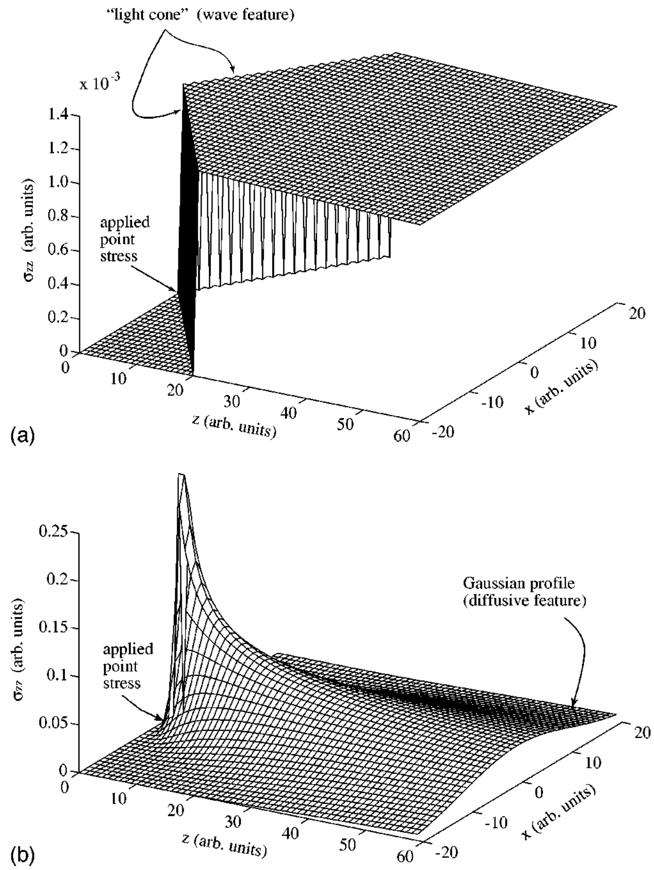

FIG. 1. Unification of (a) the wave limit representative of the theory of Bouchaud *et al.* [8], and (b) the diffusive limit representative of the theory of Liu *et al.* [11] shown through plots of the stress distribution in an unbounded medium for an applied δ-function stress. Both the ''light cones'' of the wave limit and the Gaussian profile of the diffusive limit are seen. Parameters are $c=1$, $\alpha=0.005$ in (a), and $D=1$ (along with $\alpha\to\infty$, $c\to\infty$, $c^2/\alpha=D$) in (b). Units are arbitrary.

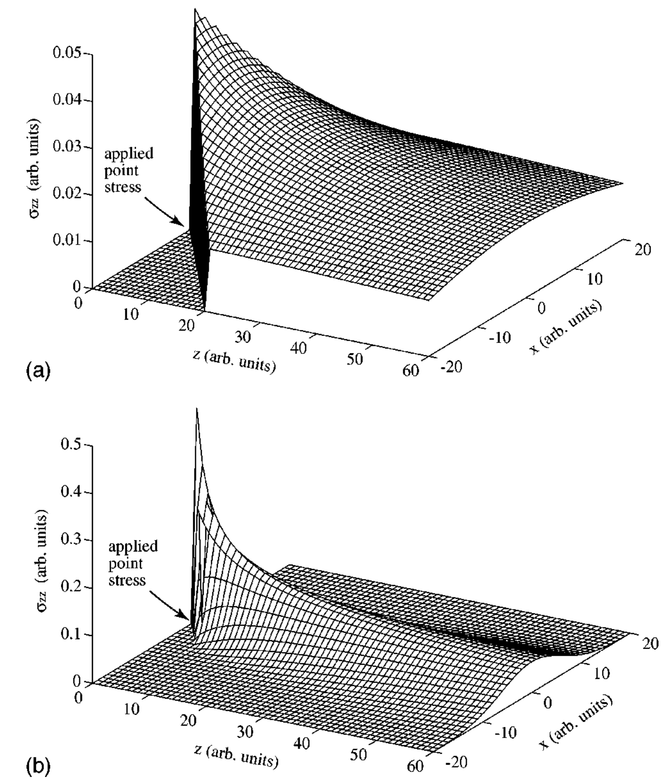

FIG. 2. Stress distribution in an unbounded medium for an applied δ-function stress for intermediate parameter ranges. Parameters are $c=1$, $\alpha=0.2$ in (a), and $c=1$, $\alpha=3.2$ in (b). Units are arbitrary.

$$\sigma_{zz}(x,z)=(1/2)[\delta(x+cz)+\delta(x-cz)] \tag{5.5}$$

as in Ref. [8]. Our theory shows that, in addition, there is a nonvanishing stress distribution *within* the light cones. This stress is given by our term T. In the limit that reduces our theory to the opposite extreme of Liu *et al.* [11], the light cones spread out to coincide with the surface $z=0$, and the entire region experiences stress:

$$\sigma_{zz}(x,z)=\frac{e^{-x^2/4Dz}}{(4\pi Dz)^{1/2}}. \tag{5.6}$$

Needless to say, the solution for arbitrary prescribed distribution at the surface $z=0$ is obtained by using the solution (5.3) as a propagator, i.e., as

$$\sigma_{zz}(x,z)=\int dx' G(x-x',z)\sigma_{zz}(x',0), \tag{5.7}$$

where $G(x,z)$ is the right hand side of Eq. (5.3).

In Figs. 1 and 2, we display plots for the ''normal stress'' $\sigma_{zz}(x,z)$ in arbitrary units as predicted from the present analysis when the applied stress is a δ function. The extreme limits are shown in Fig. 1 and intermediate cases are shown in Fig. 2. In all cases except that depicted in Fig. 1(b), $c=1$, and the quantity plotted corresponds to just the wake portion $e^{-\alpha z/2}T$ in Eq. (5.3), viz., it does not include the singular part of the stress. In Fig. 1(b), the quantity plotted is the entire stress given by Eq. (5.6) and $D=1$. The wave extreme can be seen in Fig. 1(a) for which α is nearly vanishing: $\alpha=0.005$, and corresponds essentially to the wave limit of Bouchaud *et al.* [8]. The ''light cone'' behavior mentioned by those authors is evident in Fig. 1(a). Case 1(b) is the extreme diffusive limit as would correspond to the equations of Liu *et al.* [11]. Figure 2 describes intermediate situations inaccessible to extreme wave or diffusive treatments. We see both ''light cone behavior'' and the onset of the diffusive (Gaussian) profile further down from the surface. Parameter values in Figs. 2(a) and 2(b) are $\alpha=0.2$, $\alpha=3.2$, respectively. The scale changes along the stress axis are indicative of the fact that the wake is zero for vanishing α, and rises in value as the evolution becomes more diffusive.

B. Stress distribution in pipes

To analyze the distribution of stress in long pipes we solve the boundary value problem relevant to Eq. (5.2). The application of the method of separation of variables is straightforward as far as the form of the solution is concerned,

$$\sigma_{zz}(x,z)=\sum_k (A_k\cos kx+B_k\sin kx)g_k(z); \tag{5.8}$$

$$g_k(z)=e^{-(\alpha/2)z}\left[\cosh \Omega_k z+\frac{\alpha}{2\Omega_k}\sinh \Omega_k z\right],$$

$$\Omega_k=\sqrt{\alpha^2/4-c^2k^2}. \tag{5.9}$$

However, the choice of the boundary condition to be used is almost as difficult to motivate on physical grounds as the constitutive relations. For illustrative purposes only, we will take the stress to be vanishing on the boundaries of the compact. Thus, we assume that the compact extends from $x=-L/2$ to $x=L/2$, and that $\sigma_{zz}(\pm L/2,z)=0$. Only the cosines in Eq. (5.8) survive as a result of the obvious symmetry in the problem, the A's are obtained from the functional form of the applied stress at $z=0$:

$$A_k=(2/L)\int_{-L/2}^{L/2}dx\ \cos kx\,\sigma_{zz}(x,0), \tag{5.10}$$

and, with $m=0,1,2,\ldots,$

$$k=(2m+1)(\pi/L). \tag{5.11}$$

For the usual case wherein a constant punch pressure p_0 is applied across the top surface of the compact,

$$\sigma_{zz}(x,z)=p_0e^{-(\alpha/2)z}\sum_{m=0,1,\ldots}\frac{4(-1)^m}{\pi(2m+1)}\times\cos\frac{(2m+1)\pi x}{L}\left[\cosh(\omega_m z)+\frac{\alpha}{2\omega_m}\sinh(\omega_m z)\right], \tag{5.12}$$

$$\omega_m=\{\alpha^2/4-[(2m+1)(c\pi/L)]^2\}^{1/2}. \tag{5.13}$$

Equation (5.12) shows that the stress dependence on the horizontal coordinate x is oscillatory as is appropriate to the boundary stress being held constant throughout the pipe walls. The dependence along the vertical coordinate is hyperbolic if $\alpha/2>2c\pi/L$. When this inequality is not satisfied, the dependence is hyperbolic (trigonometric) for modes for which $\alpha/2$ is greater (smaller) than $(2m+1)(c\pi/L)$. Our analysis thus reveals the interesting feature that the nature of the stress variation along the *vertical* coordinate depends on the mode considered, while which modes predominate is determined by the top surface stress distribution along the *horizontal* coordinate. Here, we see similarities with, and departures from, the behavior predicted by Janssen [30] and Thompson [24]. Their calculations show the stress as having a simple exponential dependence on depth. This could result from the retention of only the lowest spatial frequency terms in the Fourier sum in Eq. (5.12). If, however, α is relatively small, i.e., if the spatial memory function that decays slowly, a significant contribution to the behavior of the stress can be made by the higher frequency components. This can result in oscillations in the stress as a function of depth.

The analysis of Thompson [24] predicts that the stress variation along the center line of the compact is constant. This is in contradiction with experimental observations. It is interesting to examine this center line variation with the help of our results. Equation (5.12) gives, for the center line stress,

$$\sigma_{zz}(0,z)=p_0e^{-(\alpha/2)z}\sum_{m=0,1,\ldots}\frac{4(-1)^m}{\pi(2m+1)}\left[\cosh(\omega_m z)+\frac{\alpha}{2\omega_m}\sinh(\omega_m z)\right]. \tag{5.14}$$

The sum can be evaluated exactly by going into the Laplace domain. Using tildes to denote the Laplace transform, ε being the Laplace variable, we have

$$\widetilde{\sigma_{zz}}(0,\varepsilon)/p_0=\sum_{m=0,1,\ldots}\frac{4(-1)^m}{\pi(2m+1)}\times\frac{\varepsilon+\alpha}{\varepsilon^2+\varepsilon\alpha+[(2m+1)c\pi/L]^2}=\frac{1}{\varepsilon}\left[1-\mathrm{sech}\left(\frac{L}{2c}\sqrt{\varepsilon^2+\varepsilon\alpha}\right)\right]. \tag{5.15}$$

The summation is confirmed in standard tables [34]. The Laplace inverse of the extreme right side of Eq. (5.15) is known for $\alpha=0$. In that wave limit, the stress is a square wave $W(z)$ along the z coordinate. It is constant at the applied value p_0 for $0<z<L/2c$, flips to $-p_0$ for $L/2c<z<3L/2c$, flips back to p_0 for $3L/2c<z<5L/2c$, and continues alternating in this fashion. Using a theorem [35] that allows one to calculate the Laplace inverse of $\tilde{f}(\sqrt{\varepsilon^2-a^2})$ if the Laplace inverse of $\tilde{f}(\varepsilon)$ is known, it is possible to invert Eq. (5.15) to obtain the center line stress explicitly as

$$\sigma_{zz}(0,z)/p_0=1+\int_0^z du\,e^{-(\alpha/2)u}\left[M(u)+(\alpha/2)\times\int_0^u ds\,I_1(s)M(\sqrt{u^2-s^2})\right], \tag{5.16}$$

where I_1 is a modified Bessel function and $M(z)$, the derivative $dW(z)/dz$ of the square wave $W(z)$ described above, can be expressed as an infinite sum of δ functions centered at multiples of $L/2c$. In the completely diffusive limit, the center line stress distribution is given by $(1/\varepsilon)\{1-\mathrm{sech}\,[(L/2)\sqrt{\varepsilon/D}]\}$ in the Laplace domain. Inversion yields

$$\sigma_{zz}(0,z)/p_0=2\int_0^{1/2}d\nu\ \ \theta_1\left(\nu\left|\frac{4Dz}{L^2}\right.\right), \tag{5.17}$$

where θ_1 is the elliptic theta function of the first kind.

In Fig. 3 we plot the center line stress as a function of depth z for $c=1$. Several features are worthy of note. In (a) the stress is plotted for three values of α: 6, 12, and 24. The discontinuities near the values of multiples of $L/2c$, which are evident particularly for the smaller values of α, arise from wall reflections and are a consequence of the finiteness of the ''speed parameter'' c. The stress is constant from the surface to the depth that corresponds to the first reflection.

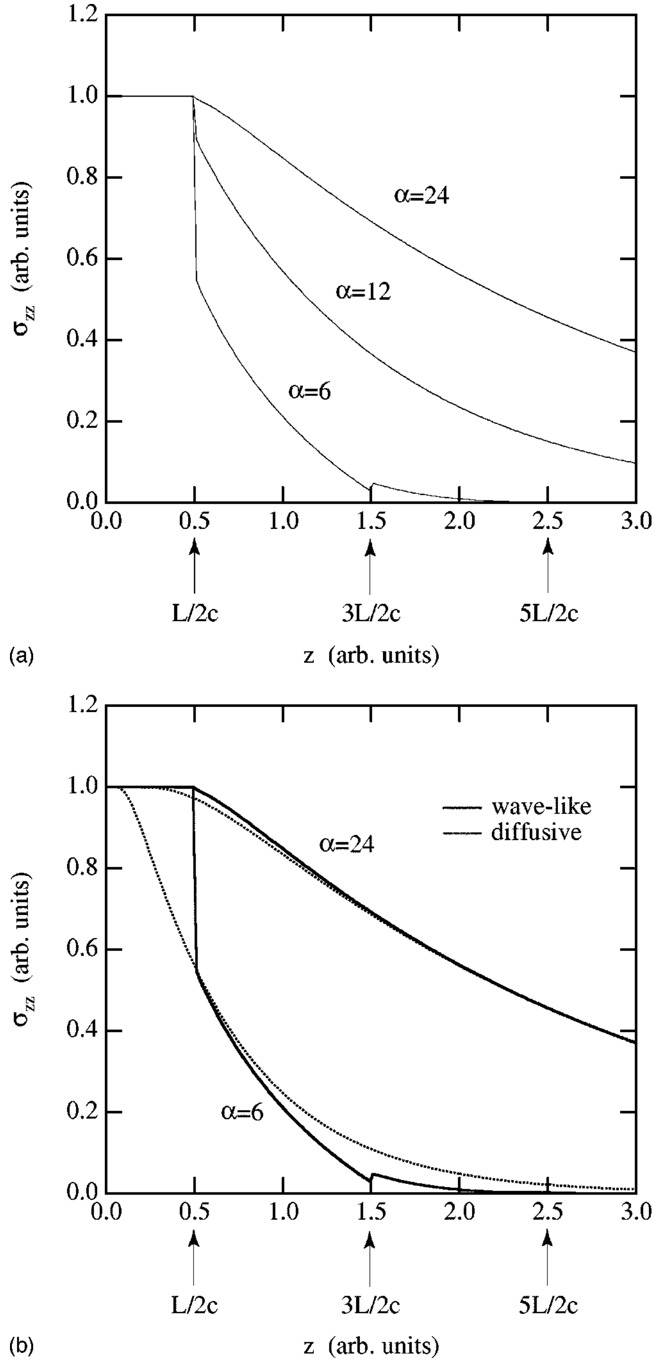

FIG. 3. The center line stress plotted as a function of depth z for $c=1$, $L=1$ from equation (5.16). Units are arbitrary. In (a) the stress is plotted for three values of α: 6, 12, and 24. In (b) the solutions of the telegrapher's equation are shown by solid lines and those of the corresponding diffusion equation (see text) by dotted lines. Thompson's results [24] that the center line stress is independent of depth could correspond to the extreme diffusive limit.

The decay at greater depths is slower for larger values of α. In (b) we show a comparison of the telegrapher's equation analysis to the diffusion equation analysis (as would be appropriate to Ref. [11]). Diffusion results are shown by dotted lines, the telegrapher's results by solid lines and values of α chosen are 6 and 24, respectively. The parameter in the diffusion equation, D, is taken to be related to the telegrapher's equation parameters c,α through $D=c^2/\alpha$. The difference is more pronounced for lower values of α and the diffusion equation results do not exhibit discontinuities. Our theoretical prediction that the center line stress generally depends on z is in agreement with experiment but in conflict with the theory of Thompson [24], which could correspond to the extreme diffusive limit.

Our purpose in presenting the above treatment of the boundary value problem has been only illustrative. Wave features present in the wave or telegrapher's equation can lead to unphysical consequences such as negative values for the stress particularly as a consequence of imposed boundary conditions. A more general treatment that eliminates these features has been developed in the following paper [36] where we have compared our predictions to experiment.

Boundary value treatments of the kind we have given here appear not to have been presented from Bouchaud *et al.*'s wave equation (4.2), or from Liu *et al.*'s diffusion equation (4.3). The former authors have focused attention on what may be termed ''the ray optics limit'' of the wave equation in their treatment of silo geometry, while the latter authors have concentrated on a mean field treatment of the stress distribution. Our boundary condition analysis should complement these earlier methods of investigation.

VI. NONLINEAR EVOLUTION EQUATIONS

Our analysis thus far has used the reduced form of the ''continuity equation'' (2.6) in which the term ρg is considered negligible. This is warranted when the stress due to the weight of the granular material is a small perturbation on the applied stress—a situation that is usual during compaction. However, we return to the general case in this section and show how the inclusion of the source term ρg leads to nonlinear evolution equations.

Our point of departure is the full form of Eq. (2.12) with the gravity term added

$$\frac{\partial\sigma_{zz}(x,y,z')}{\partial z}=D\int_0^z dz'\,\phi(z-z')\nabla^2\sigma_{zz}(x,y,z')+g\rho(z). \tag{6.1}$$

Study of the stress-density relation in powder compacts has led to forms for the equation of state that expresses the density ρ as an explicit nonlinear function of the stress component σ_{zz} [26,27]. If one recognizes that the z dependence of ρ in Eq. (6.1) occurs through such an explicit stress dependence of ρ, one sees Eq. (6.1) to be an integrodifferential equation of the Volterra type with a nonlinear forcing term

$$\frac{\partial\sigma_{zz}(x,y,z')}{\partial z}=D\int_0^z dz'\,\phi(z-z')\nabla^2\sigma_{zz}(x,y,z')+g\rho(\sigma_{zz}). \tag{6.2}$$

For an exponential ''memory'' $\phi(z)=\alpha\exp(-\alpha z)$, the nonlinear equation takes the form

$$\frac{\partial^2\sigma_{zz}}{\partial z^2}+\left[\alpha-g\frac{d\rho(\sigma_{zz})}{d\sigma_{zz}}\right]\frac{\partial\sigma_{zz}}{\partial z}=c^2\nabla^2\sigma_{zz}+\alpha g\rho(\sigma_{zz}), \tag{6.3}$$

which is a nonlinearly driven telegrapher's equation with a nonlinear damping constant. The limits for extreme forms of the ''memory'' are the nonlinearly driven wave equation

$$\frac{\partial^2\sigma_{zz}}{\partial z^2}-g\left[\frac{d\rho(\sigma_{zz})}{d\sigma_{zz}}\right]\frac{\partial\sigma_{zz}}{\partial z}=c^2\nabla^2\sigma_{zz}, \qquad (6.4)$$

and the nonlinearly driven diffusion equation

$$\frac{\partial\sigma_{zz}}{\partial z}=D\nabla^2\sigma_{zz}+g\rho(\sigma_{zz}), \qquad (6.5)$$

respectively. As an example, we state the evolution equation that results in the extreme diffusive limit by using the equation of state given by Kenkre *et al.* [27]:

$$\frac{\partial\sigma_{zz}}{\partial z}-g\left\{\rho_0\left[1-\left(\frac{\rho_\infty-\rho_0}{\rho_\infty}\right)[b_1(1-e^{-\sigma_{zz}/\sigma_l}) +b_2e^{-\sigma_a/\sigma_{zz}}]\right]^{-1}\right\}=D\frac{\partial^2\sigma_{zz}}{\partial x^2}. \qquad (6.6)$$

We have studied equations such as (6.6) through analytical approximations including perturbation techniques and mode-coupling procedures as well as via numerical methods. The interplay of the diffusive evolution with the nonlinearities arising from the rearrangement process associated with σ_l and the crushing process associated with σ_a gives rise to rich behavior that will be reported elsewhere. These considerations are of importance to self-compacting systems such as sandpiles or unconsolidated geological features such as hillsides and mine tailings.

VII. REMARKS

We discuss below the primary steps that constitute our theory, the advantages and disadvantages of our theoretical framework, and work in progress. The essential ingredients of the theory we have presented in this paper are as follows: (i) the treatment of granular material through the Cauchy relations (2.3), (2.4), (2.5) for the stress tensor under the assumption of the validity of continuum mechanics, (ii) interpretation of one of the Cauchy relations as a two-dimensional continuity equation (2.6) for a fluid whose density and flux are given respectively by σ_{zz} and a two-dimensional vector whose x and y components are σ_{xz} and σ_{yz}, (iii) a search for additional relations between the other stress components in the form of a constitutive relation for the flow of the ''fluid,'' and a statement of such a new constitutive relation, (2.9) or (3.7), obtained through a *linear* extension of previous arguments of Janssen-Thompson-Bouchaud, (iv) the derivation of a generally non-Markoffian evolution equation (4.1) for σ_{zz} involving a spatial memory function with the related interpretation of the vertical coordinate as a time variable, (v) unification of two quite unrelated treatments in the literature by deriving them as two extreme cases of our general result: the wave equation (4.2) and the diffusion equation (4.3), (vi) application of our telegrapher's equation analysis to treat stress distributions in unbounded as well as bounded granular media, which add interesting features to the results of earlier analyses and suggest entirely new features not accessible to earlier theory, (vii) derivation of nonlinear evolution equations consisting of the telegrapher's (or memory) equation driven by terms that combine the present analysis of stress gradients with the previous analysis of the stress-density equation of state given by some of the present authors [27].

One of the shortcomings of our theory, which it shares with all treatments known to us in the literature, is that the exact physical origin or justification of the constitutive relations is not known. Since all the treatments justify the constitutive relations only on mathematical grounds, there is certainly a possibility of the emergence of spurious results. We know of no solution to this problem at the present. The validity of the assumptions can be judged only through comparison to experiment. Another point of concern is related to the fact that, whereas the diffusion equation preserves positivity of the quantity it governs, the wave and the telegrapher's equations do not. The treatment of Bouchaud *et al.* [8] as well as the telegraphers equation analysis can, therefore, give rise to unphysical negativities. These undesirable features appear in two dimensions for all media and one dimension for bounded media under certain boundary conditions. An extension of our analysis, which removes these negativities in practical applications to experiment, is given in the succeeding paper [36]. An additional problem that our theory shares with all treatments that interpret the z coordinate as a time coordinate (and they include the descriptions of both Bouchaud *et al.* [8] and Liu *et al.* [11]) is that, in the present form, they are valid only in long pipes or media without a bottom. Termination in the z direction as in a compact introduces ''boundary conditions in time,'' which appear difficult to treat from evolution equations. In the true time evolution situation, we predict behavior at a later time, given spatial boundary conditions for all time and an initial condition. The incorporation of a ''final'' condition, i.e., a boundary condition at large values of time seems difficult to implement. Work is under way on this conceptual (technical) problem. Pending the resolution of this feature, we must consider the present theories in this category, whether ours or those of Refs. [8] and [11], to be restricted in their validity to predictions in regions far away from bottom surfaces.

The success of our theory lies in the unification of the seemingly disparate treatments of Refs. [8] and [11], the explicit application to existing observations with both qualitative and quantitative agreement [36], and the considerable potential for further results. We briefly mention this potential by alluding to our ongoing work.

We have seen in Sec. V that explicit incorporation of finite acceleration due to gravity leads to nonlinear equations of stress distribution. We are currently involved in analytical approximations and numerical investigations of the solution of these nonlinear equations. We point out that these nonlinearities arise from the combination of our framework with stress-density equations of state current in the literature rather than from generalizations of constitutive relations as in [8]. Our theory allows us to do ''smart processing,'' i.e., achieving desired stress distributions in given locations by controlling the applied stress. By calculating the Fourier coefficients for a system whose extent changes in width, it is possible to analyze the compaction behavior of constrained granular systems beyond that of pipes. Such systems include funnel-shaped silos, dies with sloping sides, and perhaps even sandpiles wherein the boundaries are defined by the sides of the pile that have relaxed to the angle of repose. Because we treat the z-coordinate formally as time, this scenario is simply the problem of calculating the transport behavior in a system with moving boundaries. We hope to report on these and related matters in forthcoming publications.

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